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A point explosion in an arbitrary atmosphere

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It is shown that the method developed by Laumbach & Probstein (1969) for an exponential atmosphere can be extended to the case of an arbitrary atmospheric density distribution. Results are given for various burst-point altitudes in a model atmosphere and compared with those corresponding to an exponential atmosphere.

1. Introduction

It is shown that the motion of the strong shock generated by an intense explosion can be determined for an atmosphere of arbitrary density distribution by using a simple extension of the method developed by Laumbach and Probstein (1969) for an exponential atmosphere. The solution depends upon the altitude of the burst point but can still be scaled for an arbitrary explosion energy. Also, the shape of the shock front at a given instant of time can still be determined from integration of the differential equations of the model along the vertical. We give results corresponding to explosions occurring between 10 and 190 km for (i) the *U.S. Standard Atmosphere* (U.S. Government Printing Office, Washington, 1962) and (ii) exponential atmospheres which approximate this standard atmosphere at the burst point.

2. Equations of motion

We recall the basic assumptions of the model presented in Laumbach & Probstein (1969): (*a*) the shock wave is sufficiently strong that counterpressure may be neglected and the strong shock relations can be applied; (*b*) the gas is considered to be a calorically and thermally perfect one characterized by an adiabatic exponent γ; (*c*) body forces due to the earth's gravitational and magnetic fields, wind effects and heat transfer by radiation and conduction are neglected; (*d*) the atmosphere is thus considered to be initially at rest and cold, i.e. at zero temperature and pressure; (*e*) the entropy of a fluid particle remains constant after passage through the shock. In order to be able to solve at least approximately the partial differential equations of this model it is further assumed that the flow field is 'locally radial'.

The geometry of the flow is as sketched in figure 1, where r is the Eulerian coordinate of a fluid particle of thickness dr, $R(t,\theta)$ is the position of the shock front at time t for a polar angle θ, and $h(t,\theta) = R(t,\theta)\cos\theta$ is the corresponding altitude as measured from the burst point O. The assumption of 'local radiality' then amounts to neglecting gradients in the θ direction or, equivalently, to considering the streamlines from the burst point O as straight. Clearly, under the preceding assumptions the problem is axisymmetric about the vertical axis through the burst point.

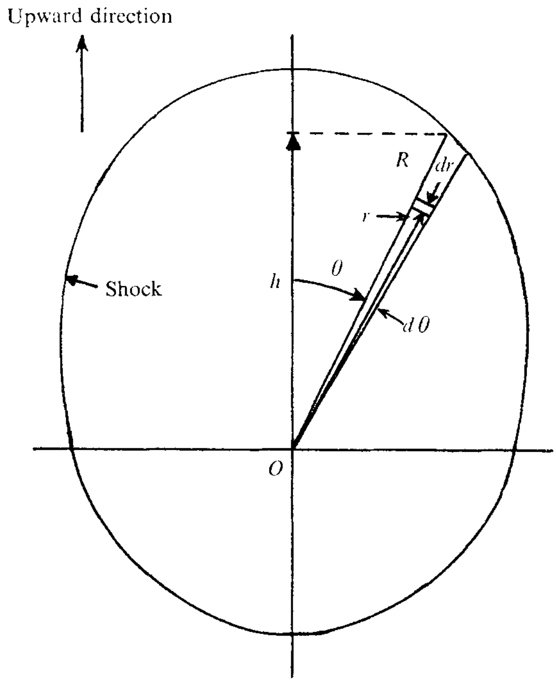

FIGURE 1. Flow geometry.

The method of solution used in Laumbach & Probstein (1969) amounts to deriving an ordinary differential equation for R from the integral energy conservation equation

$$\frac{E}{4\pi} = \int_0^R \frac{p}{\gamma - 1} r^2\, dr + \int_0^R \frac{1}{2}\left(\frac{\partial r}{\partial t}\right)^2 \rho_0 r_0^2 dr_0, \tag{2.1}$$

where E is the total hydrodynamic energy of the flow field and is considered to be known and constant, $p = p(r_0, t; \theta)$ is the pressure at time t, r_0 being defined as the position of a particular fluid particle at the burst time $t = 0$, and, finally, $\rho_0 = \rho_0(r_0 \cos\theta)$ is the initial density distribution.

In (2.1) the pressure is evaluated from the momentum equation in integral form, namely

$$p(r_0, t; \theta) - p_s(R; \theta) = \int_{r_0}^R \frac{1}{r^2}\frac{\partial^2 r}{\partial t^2} \rho_0 \bar{r}_0^2 d\bar{r}_0, \tag{2.2}$$

where from the strong shock assumption the pressure behind the shock front is given by

$$p_s = \frac{2}{\gamma + 1} \rho_0 \dot{R}^2, \tag{2.3}$$

where the dot implies differentiation with respect to t.

By following the same steps as in Laumbach & Probstein (1969), but for an arbitrary density distribution $\rho_0(h)$, we obtain a differential equation in R and t from which there is easily derived an equation in $h = R\cos\theta$ and a new time scale

$$\tau = t[E|\cos^5\theta|/4\pi]^{\frac{1}{2}}, \tag{2.4}$$

namely

$$F(h)\frac{d^2h}{d\tau^2}+G(h)\left(\frac{dh}{d\tau}\right)^2 = \alpha, \tag{2.5}$$

where

$$\alpha = \begin{cases} 1 & \text{for} \quad 0 \leqslant \theta < \tfrac{1}{2}\pi, \\ -1 & \text{for} \quad \tfrac{1}{2}\pi < \theta \leqslant \pi, \end{cases} \tag{2.6}$$

$$F(h) \equiv \frac{4}{3}\frac{(2\gamma-1)\,h}{(\gamma-1)\,(\gamma+1)^2}\phi(h), \tag{2.7}$$

$$G(h) \equiv \frac{2}{3}\frac{h^3\rho_0(h)}{(\gamma-1)\,(\gamma+1)}+\frac{2}{3}\left[\frac{h}{(\gamma+1)^2}\left(\frac{d(\ln\rho_0)}{dh}\right)_h+\frac{7\gamma+3}{(\gamma+1)^3}\right]\phi(h), \tag{2.8}$$

with

$$\phi(h) = \int_0^h \rho_0(\bar{h})\,\bar{h}^2\,d\bar{h}. \tag{2.9}$$

The integration is conveniently performed once (2.5) and (2.9) have been cast into the equivalent system of three simultaneous differential equations

$$dh/d\tau = u, \tag{2.10a}$$

$$du/d\tau = (\alpha - G(h)\,u^2)/F(h), \tag{2.10b}$$

$$d\phi/d\tau = \rho_0(h)\,h^2u. \tag{2.10c}$$

It remains to choose proper initial conditions for the determination of the solution of this system.

For a uniform atmosphere of density $\rho_B = \rho_0(0)$, equation (2.5) reduces to

$$\frac{4}{9}\frac{(2\gamma-1)\,\rho_B}{(\gamma-1)\,(\gamma+1)^2}\bar{h}^4\frac{d^2\bar{h}}{d\tau^2}+\frac{4}{9}\frac{\gamma(5\gamma+1)\,\rho_B}{(\gamma-1)\,(\gamma+1)^3}\bar{h}^3\left(\frac{d\bar{h}}{d\tau}\right)^2 = 1. \tag{2.11}$$

This equation has the first integral

$$\bar{u}^2 = \frac{\delta}{\bar{h}^3}+\frac{c}{\bar{h}^{3+\omega}}, \tag{2.12}$$

where

$$\delta = \frac{9(\gamma-1)\,(\gamma+1)^3}{2(4\gamma^2-\gamma+3)\,\rho_B} > 0, \quad \omega = \frac{4\gamma^2-\gamma+3}{(2\gamma-1)\,(\gamma+1)} > 0$$

and c is a constant of integration which remains to be determined (we are indebted to one of the referees for the following proof that c must be set equal to zero).

It is clear from the expression for $\bar{u}^2$ in (2.12) above that, if $c < 0$, then for $\bar{h} \to 0$, $\bar{u}$ becomes imaginary. Thus, we conclude that $c \geqslant 0$. However, if $c > 0$, the internal energy (the first term on the right-hand side of (2.1)) approaches $-\infty$ as $\bar{h} \to 0$, which is physically impossible since the internal energy must be non-negative. Thus, $c = 0$ and we have

$$\bar{u}^2 = \delta/\bar{h}^3. \tag{2.13}$$

On integrating this expression we obtain

$$\bar{h} = c_1 \tau^{\frac{2}{5}}, \tag{2.14}$$

where

$$c_1 = \left[\frac{225(\gamma-1)(\gamma+1)^3}{8(4\gamma^2-\gamma+3)\rho_B}\right]^{\frac{1}{5}}, \tag{2.15}$$

which in our notation is the equivalent of equation (27) of Laumbach & Probstein (1969).

3. Results

In order to illustrate the method the set of differential equations (2.10) has been integrated numerically using the *U.S. Standard Atmosphere* (1962) as a model for altitudes between sea level and 190 km. We have used the formula given in the *U.S. Standard Atmosphere Supplements* (U.S. Government Printing Office, Washington, 1966) to evaluate the corresponding density, namely

$$\rho(Z) = 1{\cdot}225/P^4(Z)\,\text{kg/m}^3, \tag{3.1}$$

where

$$P(Z) = A_0 + A_1 Z + \ldots + A_{11} Z^{11},$$

with

$$A_0 = +0{\cdot}1000000000\times 10^{1}, \quad A_1 = +0{\cdot}3393495800\times 10^{-1},$$
$$A_2 = -0{\cdot}3433553057\times 10^{-2}, \quad A_3 = +0{\cdot}5497466428\times 10^{-3},$$
$$A_4 = -0{\cdot}3228358326\times 10^{-4}, \quad A_5 = +0{\cdot}1106617734\times 10^{-5},$$
$$A_6 = -0{\cdot}2291755793\times 10^{-7}, \quad A_7 = +0{\cdot}2902146443\times 10^{-9},$$
$$A_8 = -0{\cdot}2230070938\times 10^{-11}, \quad A_9 = +0{\cdot}1010575266\times 10^{-13},$$
$$A_{10} = -0{\cdot}2482089627\times 10^{-16}, \quad A_{11} = +0{\cdot}2548769715\times 10^{-19}.$$

The relative error between the density given by this approximation and the density of the 1962 model is less than 5 %.

The quantity $d(\ln\rho_0)/dh$ appearing in the expression for $G(h)$ (equation (2.8)) was evaluated using the formula

$$d(\ln\rho_0)/dZ = -4P'(Z)/P(Z). \tag{3.2}$$

A graph of $\rho_0(Z)$ and of the density scale height $\Delta(Z) = -dZ/d(\ln\rho_0)$ is given in figure 2. The values of Δ given by the above rational approximation are accurate to within about 5 % except at very low altitudes ($\lesssim$ 1 km).

Finally, the initial values were computed using formula (2.14) for h, and

$$u = \tfrac{2}{5}h/\tau,$$

$$\phi = c_1^3\rho_B\left[\frac{1}{3}+\frac{1}{4}\left(\frac{1}{\rho}\frac{d\rho}{dh}\right)_B c_1\tau^{\frac{2}{5}}\right]\tau^{\frac{6}{5}},$$

where c_1 is as defined by (2.15). The value of τ used in these formulae was determined so that the relative difference between $du/d\tau$ as given by (2.10*b*) and its corresponding value $du/d\tau = -\frac{6}{25}h/\tau^2$ for the uniform atmosphere was less than 1 %.

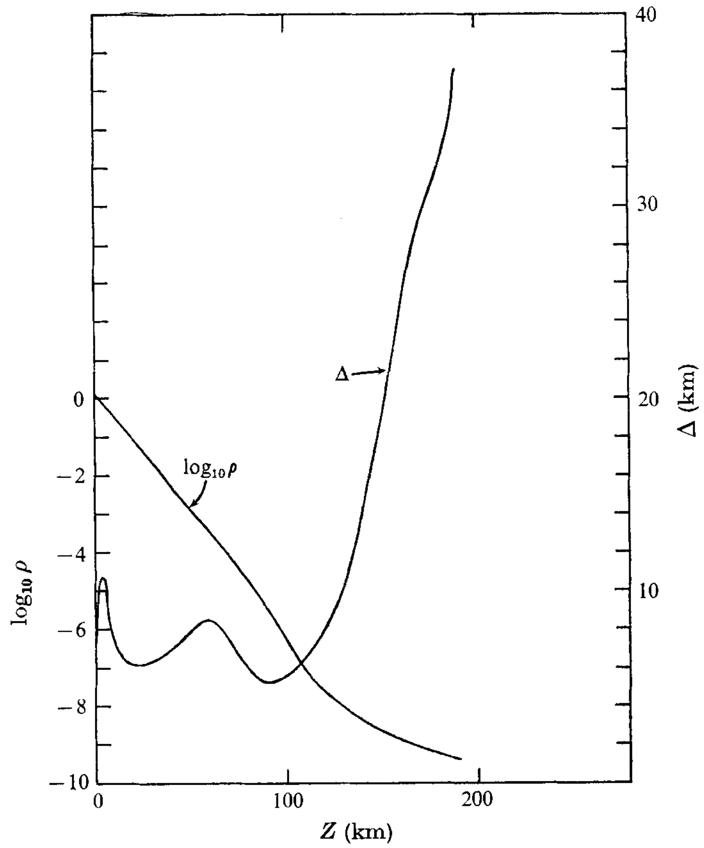

FIGURE 2. Common logarithm of the density (in kg m^{-3}) and density scale height (in km) profiles used for numerical computations (source: *U.S. Standard Atmosphere Supplements*, 1966). Note that the peak in Δ at $\sim$ 4 km is not physical and is probably due to 'end-point' inaccuracies in the approximation used.

The results of the computation are shown in figures 3 and 4 for $\gamma = 1{\cdot}4$ and explosions occurring at altitudes $Z_B = 10$, 20, 30, 40, 50, 100, 150 and 190 km. Only the quantities $Z = Z_B + h$ and h' $(= dh/d\tau)$ have been plotted versus τ since the purpose of these calculations was essentially illustrative. We also show in these figures the results corresponding to an atmosphere whose density varies exponentially with altitude according to the formula $\rho(h) = \rho_B \exp(-h/\Delta_B)$, where Δ_B is the value of Δ at the burst point as derived from the rational approximation (3.2).

Comparison of the respective sets of curves indicates that the oscillations in Δ in the altitude range 0–120 km have relatively little effect on the propagation of the shock starting from a burst point in this range. On the other hand, for burst altitudes in the range 120–190 km, there are considerable differences in the time evolution of the shock going downward owing to the large increase in the scale height Δ over that range.

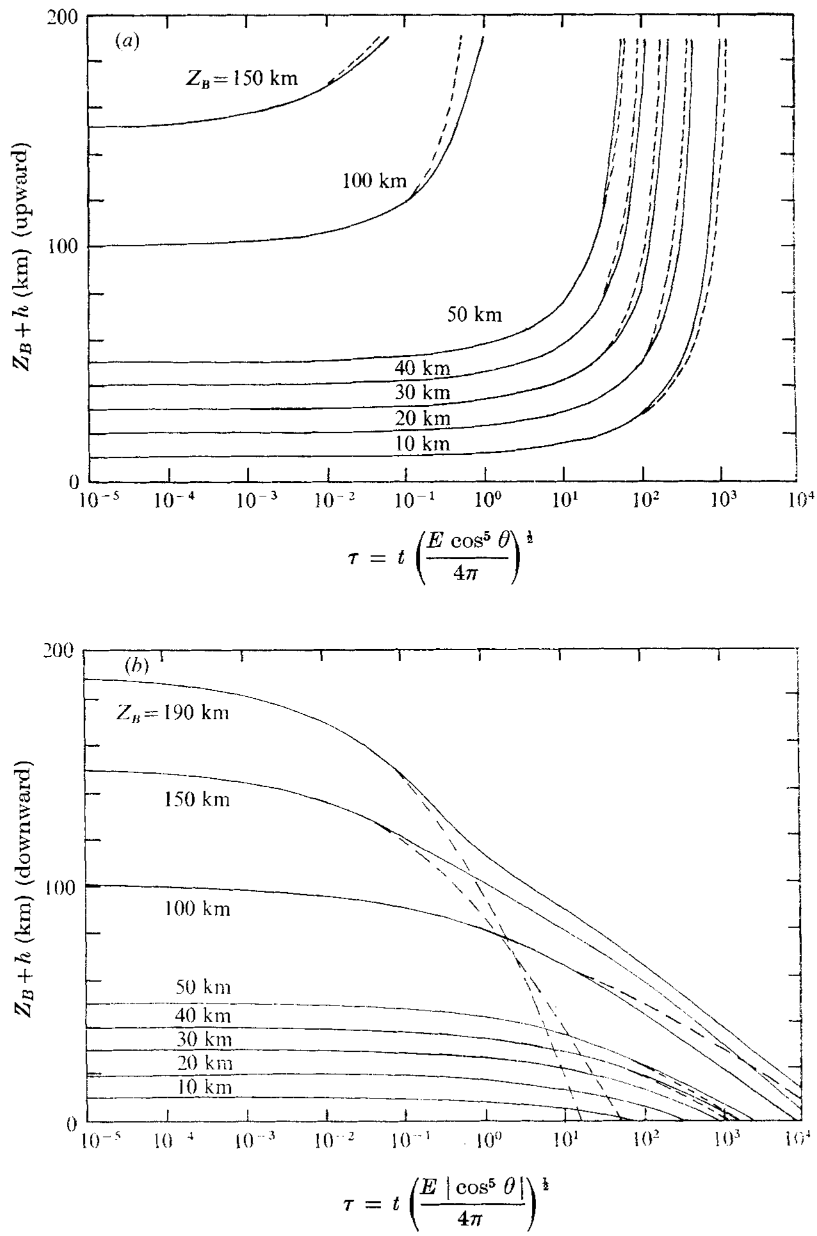

FIGURE 3. Altitude of shock front versus time in (a) the upward direction ($0 \leqslant \theta < \frac{1}{2}\pi$) and (b) the downward direction ($\frac{1}{2}\pi < \theta \leqslant \pi$). ——, *U.S. Standard Atmosphere*; – – –, exponential atmosphere.

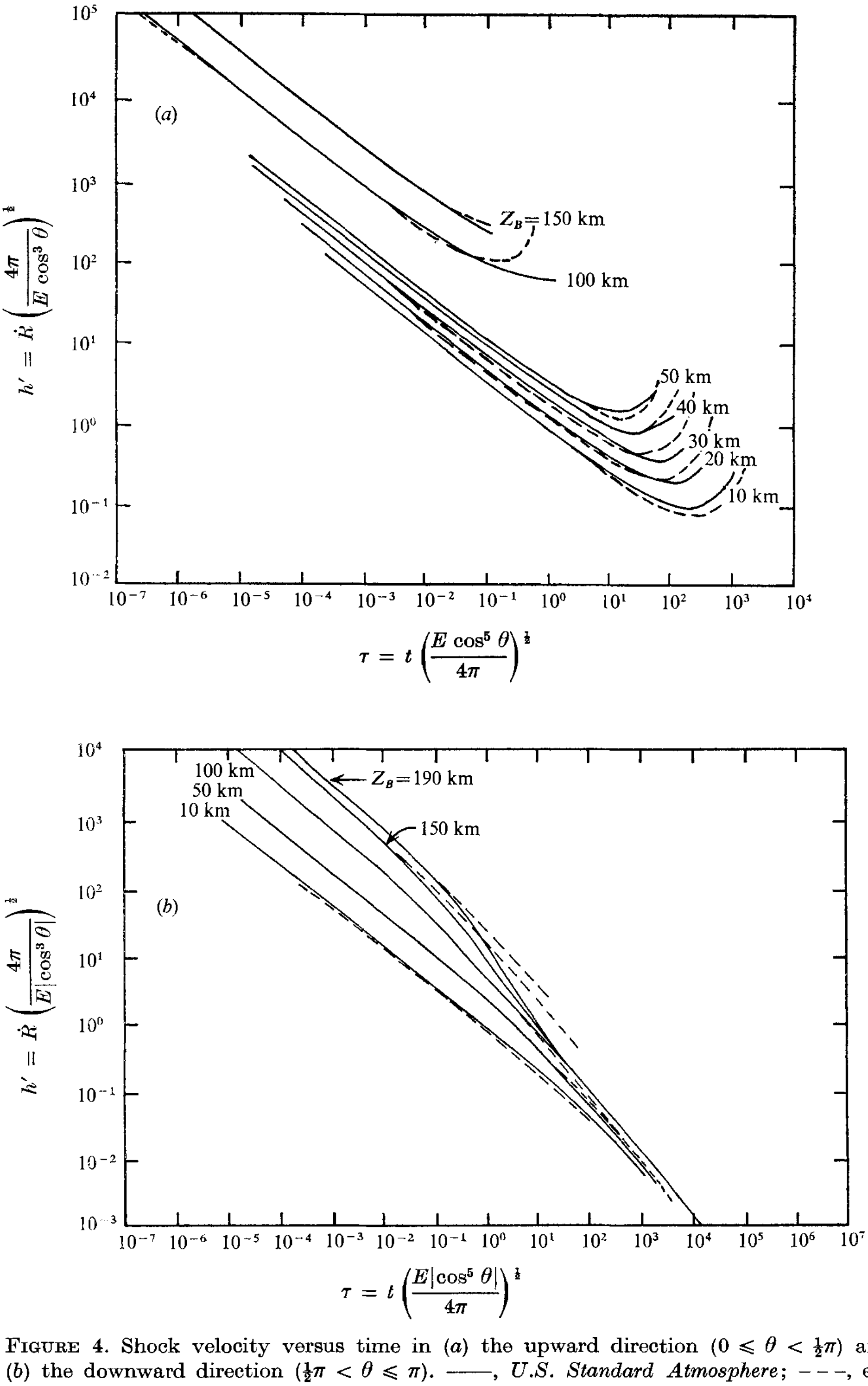

FIGURE 4. Shock velocity versus time in (*a*) the upward direction ($0 \leqslant \theta < \frac{1}{2}\pi$) and (*b*) the downward direction ($\frac{1}{2}\pi < \theta \leqslant \pi$). ——, *U.S. Standard Atmosphere*; – – –, exponential atmosphere.

4. Conclusions

It has been shown that Laumbach & Probstein's approach to the problem of a point explosion in a cold exponential atmosphere can easily be extended to atmospheres whose density distribution vary in an arbitrary fashion. The time evolution of the shock position and velocity has been computed for burst altitudes in the 0–190 km range both for the *U.S. Standard Atmosphere* and for exponential atmospheres which match this model at the burst point. A comparison of the results is presented, and from this it is possible to decide whether the exponential atmosphere description is adequate for a specific application.

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